Manufacturing and analyses of a paper cup boomerang

R. Watanabe & K. Yoshida

Tokyo Metropolitan College of Industrial Technology
Tokyo, Japan

ABSTRACT: The Department of Robotics Engineering at the Tokyo Metropolitan College of Industrial Technology has many active research projects. In the authors' laboratory, developments, experiments and analyses of various children's toys were carried out on, for example, boomerangs with three wings, PET bottle rockets, boomerang stunt planes, flying rings and boomerang dragonflies. These were developed and analysed by simple equations of motion. The experiment described in this paper involved a paper cup boomerang shaped to connect two separate paper cups. Three connected rubber bands were wrapped around the circumference at the centre of the paper cup. The top of the rubber band was held vertical to the ground. When the *cup* flew, it drew an elliptical orbit and returned to the starting point. The loci of the paper cup boomerang was analysed by using simple equations of motion. The analytical results obtained matched the experimental ones, and are presented in this paper.

INTRODUCTION

A paper cup boomerang (see Figure 1), was manufactured by connecting two paper cups. Three rubber bands were tied and wrapped around the paper cup. The experimenter's hand gripped the end point of the rubber band and held it vertical to the ground. The rubber band was pulled tightly and the paper cup released. Then, it drew an elliptical orbit, and returned to the experimenter's hand.

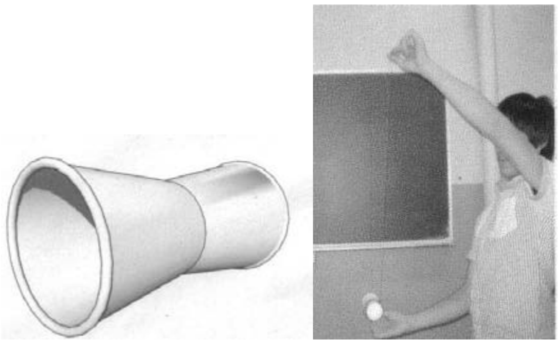


Figure 1: A paper cup boomerang and the flying method.

Figure 2: Students attend a lecture.

EQUATIONS OF MOTION

Figure 3 shows the force needed to produce a flying paper cup. Lift *L* acts vertically in the travelling direction of the paper cup. Drag *R* acts inversely in the travelling direction of the paper cup. Gravity *mg* acts in a downward direction. Lift *L* is proportional to the velocity *V* of the fluid. This is due to the *Magnus effect*.

The initial angular velocity of the paper cup is ω_0 and the initial straight velocity V_0 as shown in Figure 3. Equations of motion in the *x* and *z* axes at an angle θ are as follow:

$$m\frac{d^2x}{dt^2} = L\sin\theta - R\cos\theta \tag{1}$$

$$m\frac{d^2z}{dt^2} = -mg - L\sin\theta - R\cos\theta \tag{2}$$

Where, *m* is the mass of the boomerang paper cup. Lift *L* and drag *R* are given by the following equation:

$$L = \rho V\Gamma h = \alpha V \tag{3}$$

$$R = \frac{1}{2}C_D\rho d h V^2 = \beta V^2 \tag{4}$$

where ρ is the density of the air; Γ is circulation around the paper cup; *h* is length of the paper cup; C_D is drag coefficient; *d* is the average diameter of the paper cup and α , β are constants.

θ is the angle between path of motion and the *x* axis. The angle θ is shown as a function of time *t* by using inner product:

$$\cos\theta = \frac{\vec{v}\cdot\vec{x}}{|\vec{v}||\vec{x}|} = \frac{\dot{x}}{\sqrt{(\dot{x})^2 + (\dot{z})^2}} = \frac{k_{nx}}{\sqrt{k_{nx}^2 + k_{nz}^2}}\ , \ \sin\theta = \frac{k_{nz}}{\sqrt{k_{nx}^2 + k_{nz}^2}}$$

where, $\dot{x} = k_{nx} = v_x$, $\dot{z} = k_{nz} = v_z$,

When the above expression are substituted into equations (1),(2), they become:

$$\frac{d^2x}{dt^2} = \frac{\alpha}{m}\frac{k_{nz}}{k_{nx}}V_x - \frac{\beta}{m}\frac{\sqrt{k_{nx}^2 + k_{nz}^2}}{k_{nx}}V_x^{\ 2}$$

$$\frac{d^2z}{dt^2} = -\frac{\alpha}{m}\frac{k_{nx}}{k_{nz}}V_z - \frac{\beta}{m}\frac{\sqrt{k_{nx}^2 + k_{nz}^2}}{k_{nz}}V_z^2 - g$$

It is difficult to solve these differential equations because they are non-linear. So, the graphs of *x-t* and *z-t* are derived by numerical analysis of the Runge-Kutta method. The Runge-Kutta method is explained in Figure 4.

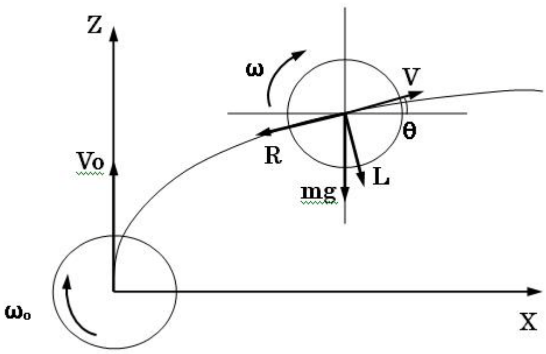


Figure 3: Coordinate axis.

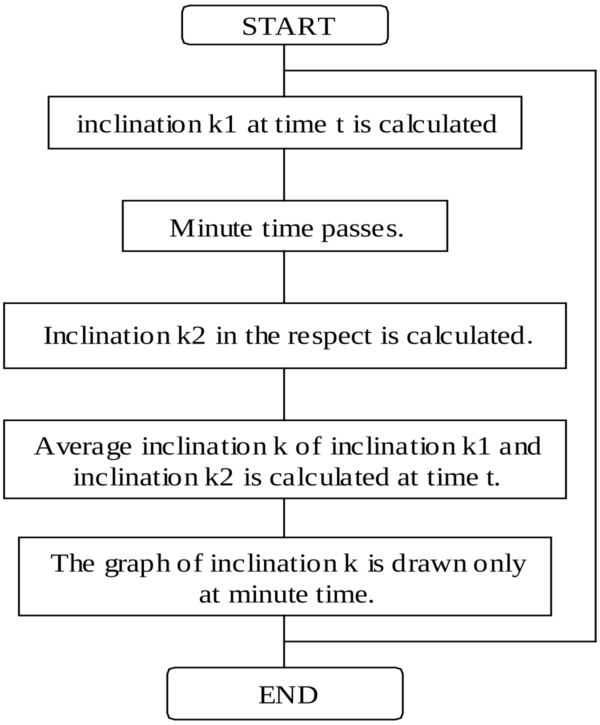


Figure 4: A flowchart of the Runge-Kutta method.

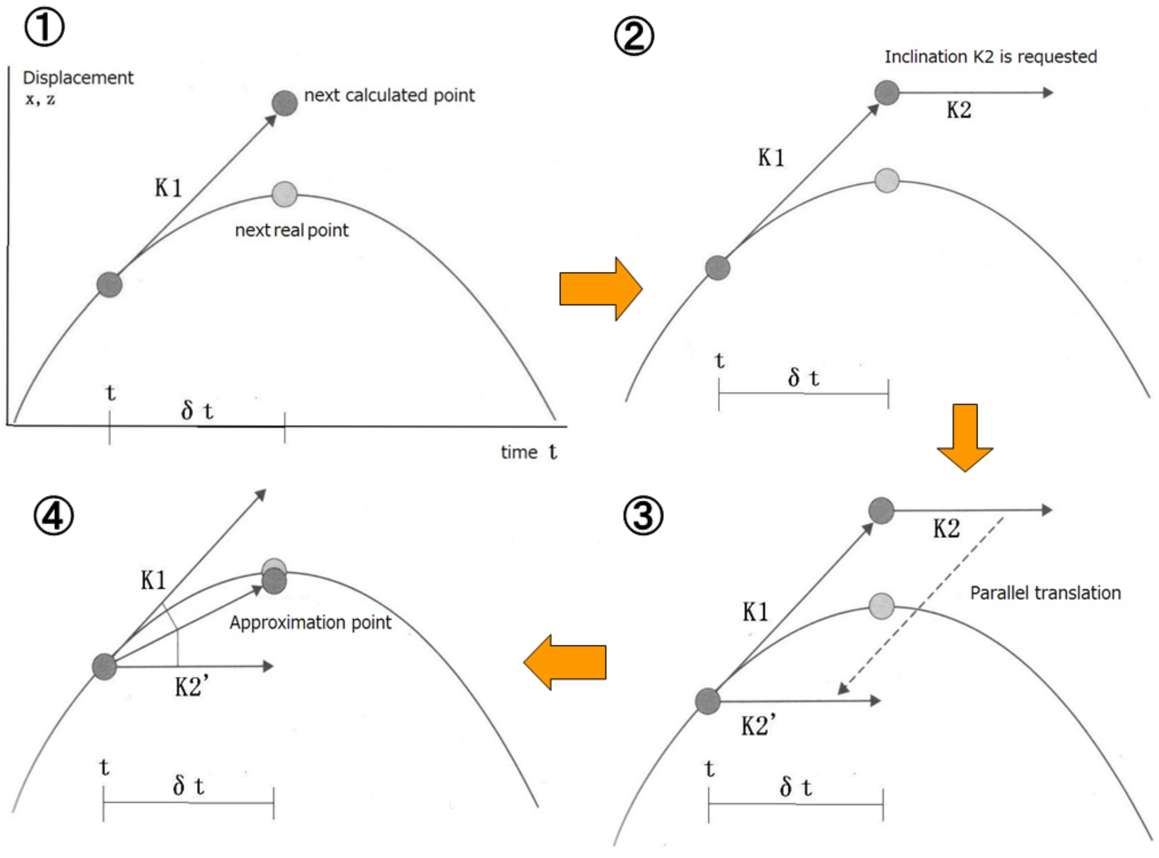


Figure 5: Graphical representation of the Runge-Kutta method.

The initial linear velocity and initial angular velocity are shown in Figure 6 using the principle of conservation of energy.

$$\frac{1}{2}k\ell^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg\ell$$

Where $v = r\omega$, $I = mr^2$

$$v = \frac{1}{2}\sqrt{\frac{4k\ell^2 + 8g\ell}{m}} \qquad \omega = \frac{1}{D}\sqrt{\frac{4k\ell^2 + 8g\ell}{m}}$$

k ：Constant of spring＝5 N/m
ℓ ：Amount of expansion of rubber ＝0.35 m
D ：Diameter of paper cup ＝0.07 m
m ：Mass＝0.015 kg
d: Average diameter of the paper cup=0.065 m
Γ : Circulation around the paper cup=0.29 m^2/s
h: Length of paper cup=0.2 m
C_D :Drag coefficient=0.68
V_0 : Initial straight velocity=4.3 m/s
ω_0 : Initial angular velocity=44 rad/s

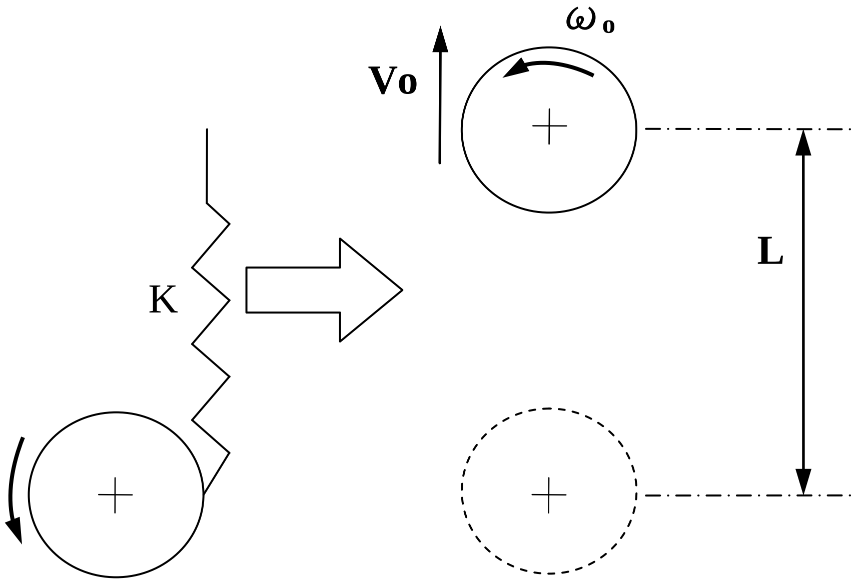
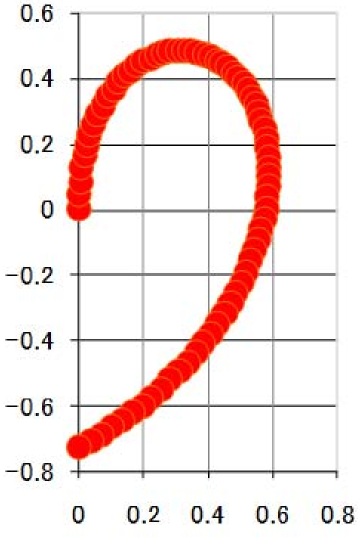


Figure 6: The principle of conservation of energy.

RESULTS

α=0.07 and β=0.0053 from expression (3),(4). The graph of the calculated results is shown in Figure 7.



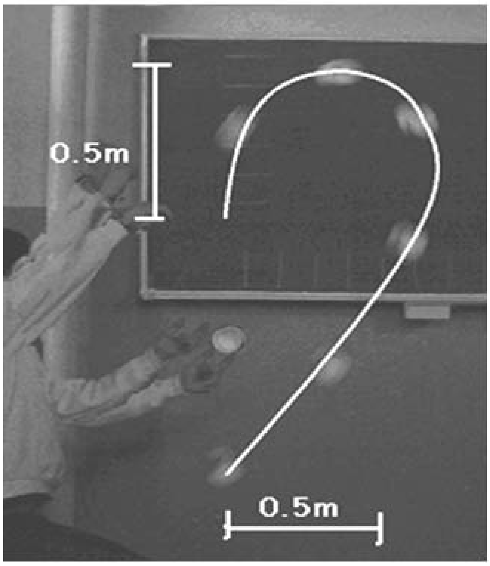


Figure 7: The analytical locus. Figure 8: The real locus.

Table1: Comparison between theoretical and real values.

	Zmax (m)	Zmin (m)	Xmax (m)	Cycle T(s)
Theory value	0.48	-0.72	0.59	0.72
Real value	0.5	-0.85	0.61	0.7
Error margin rate	4.2%	18.1%	3.4%	2.8%

The time interval for recording the paper cup was 0.1 s, because it took 0.1 s cycles of the high speed camera. This can be found in Table 1. The theoretical values were almost the same as the actual values measured, excluding the minimum value of *Z*. It was found that there were only 0.1 m error margins, when the minimum value was compared. It was thought that the angular velocity decreased because of the rubber's rolling motion.

STUDENTS' IMPRESSIONS

Students' comments were recorded as follows:

I am surprised that the paper cup returned.

When I go home, I want to redo this experiment.

I was able to experience the physics phenomenon in an entertaining way. I was interested in the physics.

CONCLUSION

The loci of a paper cup boomerang was analysed using simple equations of motion. The analyses closely match the experimental ones. Both orbits were elliptical, which have the shape of an ear. Comparison orbits, where the rubber is pulled tighter and the equipment developed using the *Magnus effect*, will be examined and planned as future tasks.