

Mathematical competencies and critical thinking in everyday life

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ABSTRACT: A different view is taken of mathematical curricula by stressing competencies in mathematical education. Critical thinking is a very general competency, which is an inherent part of mathematical and scientific thinking but which is also useful in analysing statements and events in everyday life. Mathematical education can not only contribute to problem-solving in science and engineering and to the intellect but can also make society more transparent by exposing dubious arguments. Some examples showing the interplay of mathematical curriculum, mathematical competencies, critical thinking and everyday life, are presented and discussed in this paper.

INTRODUCTION

In Germany, and countries worldwide, the relation between mathematical knowledge and mathematical competencies within the educational process is critically discussed. Indeed, there is a close connection between both concepts. Knowledge should create competencies and competencies reflect knowledge. The new trend of stressing competencies provides an opportunity to deeply investigate this relationship, and for the public to draw useful conclusions relevant to mathematical education. But some prophets overemphasise competencies; hence, leading to the competencies developed by the end of the educational process being empty or of poor content.

The mathematical curriculum contains content necessary to solve problems and to understand the interplay between various parts of mathematics. At the engineering faculty of Hochschule Wismar the curriculum is based on Linear Algebra (vectors and matrices), Analysis (calculus and differential equations), Numerical Mathematics (approximate methods, algorithmic thinking, error estimation, time and storage effort) and Stochastics (probability concepts, distributions, conclusions from random samples). Critical thinking is developed based on logical thinking, precision, error estimation and completeness of arguments. Paradoxes and counterexamples in Mathematics are especially suited for the development of such competences as critical thinking. Often they occur in the educational process as challenges to thinking. Paradoxes can play a very useful role by producing fruitful discussions, provoking deeper thinking about a subject, clarifying new concepts and giving better insight into a theory. So, they can help to remove potential conflicts between intuition, theory and reality [1][2].

Finally, modern trends in education with an emphasis on modelling, simulation, the use of software and Internet sources and project work in teams combine to facilitate the solution of more complex scientific or applied problems.

Often, interesting formulae are in the public domain. Some of these formulae can be analysed using basic mathematical means. However, they may have limited practical value since modelling assumptions may not be stated and important features may not be included in the model. Such useless mathematical formulae can endow a discipline with more importance than is justified. This is called pseudo-mathematisation. Further, news media sometimes contain mathematically derived data. The mathematically literate should be able to decide if these data are correctly used or not. In the latter case the data may be included for ideological purposes or to reflect personal bias. This is often not clear initially and may need further investigation. The topic is discussed in Schott, which includes other priorities [3]. Schott gives some insight into the interaction between mathematics and engineering in the development of competencies [4].

As an example of the use of mathematics in engineering, the calculation of support forces in engineering mechanics corresponds to the solution of systems of linear equations. Support forces turn out to be statically determinate if, and only if, there is a unique solution of the corresponding linear system. Hence, the experience of engineers is backed by a clear mathematical criterion supplying a deeper understanding. Mathematics can play a similar part in other disciplines and in everyday life.

CURRICULUM AND COMPETENCE

In earlier times, the curriculum was most important in education. The contents were fixed by study course and level. Later a catalogue of learning aims was added. At the completion of a section of a curriculum, a learner was expected to be able to solve some defined set of problems. Nowadays, the competency model is in vogue and is most favoured. It describes the abilities and skills needed for a qualification, e.g. Bachelor or Master in a certain engineering subject. There is more freedom for lecturers to fix content and the methods used to acquire the prescribed competencies. Options for further study and the solution of everyday problems are also included.

Hence, the courses are application-oriented. Besides, there are methods to measure the quality of the learning processes. But, the definition of competencies is often rather vague and open to different interpretations; the international discussion is rather broad and so the concept is used in most cases intuitively. Essential general mathematical competencies are:

- Use of *knowledge*: understanding mathematical theory, knowing important facts, linkages between disciplines;
- Correct use of *technical elements*: mathematical language, logical reasoning, rearranging or transformation of terms, use of tables, formulae, computer software and media;
- *Problem-solving*: applying and transferring solution methods, applying heuristic methods, generalising, creating new connections and concepts;
- Use of *methodology*: algorithmic, numerical, analytical and stochastic thinking, geometrical imagination;
- *Mathematical modelling*: creating suitable models, interpreting and validating results;
- *Critical thinking*: checking correctness and completeness of results;
- *Communication* skills: team working, networking.

These competencies are included in the mathematical curricula at Hochschule Wismar. The nomination of competencies establishes drawing links between subjects and competencies. Unfortunately, many students can only solve problems mechanically following given algorithms. The introduction of new teaching methods may not eliminate this phenomenon, although some experts in didactics insist it can.

PSEUDO-MATHEMATICS

There are different bases for applying mathematics in the wrong way:

- Mysticism by giving subjects a curious mathematical meaning;
- Naivety by missing competencies in the sphere of application;
- Use of mathematical models without checking the model assumptions;
- Mathematical modelling prematurely in a sphere (Quételet: Social Physics [5], Fechner: Psycho-Physics [6]);
- Trickery by upgrading a subject by linking it to mathematics;
- Rationalism (mathematics is behind all things);
- Propaganda (ideology, interests), using mathematics to influence persons' attitudes.

Quételet is considered as a starting point and milestone for the quantitative analysis of human and social qualities [5]. According to Meischner-Metge, the work of Fechner is also well regarded as the beginning of a new era in psychology [6]. Although the legitimacy of mathematics in natural sciences is recognised, the application of mathematics to the social sciences and economics is often regarded as controversial. Society is very complex and is qualitatively different from nature. Models are often either too simple or are uncritically transferred from other fields. The consequences of pseudo-mathematics are varied. On the one hand, many people believe in the omnipotence of mathematical formulae and models, and feel helpless and unable to criticise because of the power of mathematics. On the other hand, some people mistake mathematics as an elite discipline with no or only little value to the general public.

MODELLING

Often simple models are used in everyday life. If successful some people do not think of mathematics at all while others believe mathematics *regulates the world*. In any event, they take success for granted. But what happens if the result turns out to be wrong? Then, people often blame mathematics for the disaster. They do not realise the true causes of misleading results as:

- The model is too simple for the given facts.
- The model is used incorrectly.
- The range of application of the model is not respected. Prerequisites of the model are ignored.
- The model is used in a new context (without proving the legitimacy).

In everyday life, problems are not always evident. It is necessary to identify a problem and to formulate an adequate representation as a mathematical model. Many models are possible. The art is to find a sufficiently simple one, which is good enough to be applied successfully.

Science and engineering models will have a gap between the model and reality. Models show behaviour similar to real systems and, hence, can represent parts of reality under appropriate conditions. Often, models have only a restricted range of application, which must be respected to get reasonable results. Models, which hold under some conditions may have to be replaced by more general models if these conditions are violated. Sometimes, these more general models are already known; sometimes, they still have to be developed. E.g. the simple first order linear differential equation:

$$x'(t) = c \cdot x(t) \tag{1a}$$

describes exponential growth of a population or a process. Indeed, such a model is realistic if there are enough resources for a completely unconstrained development (bacteria, certain time intervals of growth processes). The British scientist Robert Malthus in 1798 assumed such a model for the human population and predicted a catastrophe on earth. In the meantime, it is known that limited resources slow down the growth of a population. Verhulst in 1838 proposed a new model with a self-limitation for large populations (bounded carrying capacity):

$$x'(t) = c \cdot x(t) - d \cdot x^2(t) \quad (c > 0, d > 0) \tag{1b}$$

The negative second term on the right side of this logistic differential equation reduces the growth to a certain limit. This equation is also of first order, but nonlinear. For d = 0, it reduces to the old model (1a). Since the logistic model was successfully applied to many growth processes in nature and economy, some believed there was a logistic principle operating in the world. But many other models with restricted growths were found, e.g. by taking account of the ageing process in a population or other additional parameters. A free harmonic oscillator is described by the linear homogenous differential equation of second order:

$$x''(t) + b \cdot x'(t) + c \cdot x(t) = 0 \quad (b \geq 0, c > 0) \tag{2a}$$

The solutions *x(t)* are undamped ($b = 0$) or damped ($b > 0$) harmonic oscillations. Often the pendulum is considered a harmonic oscillator. But the model equation for the angular displacement $x = x(t)$ is:

$$x''(t) + b \cdot x(t) + c \cdot \sin x(t) = 0 \quad (b \geq 0, c > 0) \tag{2b}$$

Hence, this is a nonlinear differential equation of second order. The solutions *x(t)* are quite different from the previous ones except for small angular displacements *x* where $\sin x \approx x$.

FRACTIONS

In physics or electrical engineering, Ohm's simple law is well known:

$$I\,[\mathrm{A}] = \frac{V\,[\mathrm{V}]}{R\,[\Omega]} \tag{3}$$

This law connects three different magnitudes, namely voltage *V*, current *I* and resistance *R* in an electrical DC circuit. It states that the current is directly proportional to the voltage (for constant resistance) and inversely proportional to the resistance. The law can be checked by experiments. For instance, the equation $V = I\,R$ shows that in a circuit with fixed resistance *R*, the measurement points (I_k, V_k) have to lie on a straight line in the *I-V* plane. Allowing for small errors in modelling and measurement, this is the case at least approximately. A more sophisticated analysis shows that the resistance depends also on the temperature *T*. So a complex model should describe the influence of *T*.

In mathematics, the sine of angle $\alpha < 90^\circ$ in a rectangular triangle is the ratio of the side (length) *a* opposite α and the hypotenuse (length) *c*:

$$\sin \alpha = \frac{a}{c}. \tag{4}$$

This relation is a definition of the sine function or at least a consequence of the definition. It can be experimentally checked by comparing the ratio of side lengths in a plotted rectangular triangle and the sine of the corresponding angle calculated by a computer. There will be an approximate equality. But, this is not a mathematical proof. It only shows that a model based on Euclidian geometry is adequate for the case of plotted triangles. By the way, fractions:

$$z = \frac{x}{y} \tag{5}$$

are also used in other fields expressing the fact or the conviction that the magnitude *z* increases, if the magnitude *x* increases and the magnitude *y* decreases. The fraction (5) is indeed the simplest model with these consequences. But the same qualitative effects can be obtained by more complex functions *f(x,y,…)* of two or more variables. So, which of these functions *f* is the most realistic one? If there are experimental data of the involved magnitudes, an optimal function *f* can be found in a defined model class using the least squares method.

There are many attempts to use fractions or more general formulae in the humanities. The Russian writer Leo Tolstoy proposed a fraction rule to measure the value *V* of man, where the power (mind, true reputation and other personal properties) determines the nominator and the meaning about itself the denominator of the fraction:

$$V = f(P,S) = \frac{P}{S} \quad \left[\frac{Power}{Self-Confidence}\right]. \tag{6}$$

According to this formula, the value increases if the power increases (at constant self-confidence) or if the self-confidence decreases (at constant power). Perhaps people believe that these statements are true or at least reasonable. But, *P* and *S* are vaguely determined. Besides, other variables should be considered, too. Perhaps, it is useless to give a formula at all. But, if people can handle mathematical fractions, the formula suggests a social message, namely that modest people with great power are of excellent moral value. Note that the message depends also on the spirit of the era. Surprisingly, such primitive formulae have influence even now. Marion Wolf, a German journalist, born in 1950, proclaimed: *...The greatness of a person is reckoned by his ability in proportion to his modesty*.

BODY MASS INDEX

Ziegler presented some mathematical topics, including the body mass index (BMI), to a broad audience [7]. This index was already introduced in 1870 by Adolphe Quételet, a Belgian mathematician and statistician. He measured the weight (mass) *m* and the height *l* of 5738 Scottish soldiers and found:

$$I = \frac{m}{l^2}\left[\frac{\mathrm{kg}}{\mathrm{m}^2}\right] \tag{7}$$

gave a good representation of a soldier's body if the following simple scale was used:

- *I* is less than 20: thin;
- *I* is between 20 and 25: normal;
- *I* is between 25 and 30: thick;
- *I* is over 30: fat.

Although this sample was not representative of all mankind, the BMI (7) is used with some variations to this day. The simple message is that everyone should reach normal weight. Many Web sites of the body care and nutrition industries offer calculators for BMI and advertise products and treatments to achieve optimal BMI values. In Germany, a person with too high a BMI has no chance to start a career as a civil servant. So the BMI is in some sense proclaimed to be a measure of health and beauty. An analysis shows that the index (7) has questionable value. Since the denominator represents a surface area according to the measure, the BMI can be interpreted as average mass load per unit surface area of the body. Does the risk to health really increase proportional to the mass and inversely proportional to the squared height (body surface)? What influences do age, sex, build or distribution of muscles and fat tissue have? Statistical investigations can help to get a more specific insight. In medicine, indeed, the scale is modified for woman, children and people with amputated body parts. Another criticism is based on geometrical arguments. Assuming that the body is approximately a cuboid with average density ρ, width $a = \alpha \cdot l$, thickness $b = \beta \cdot l$ and height $h = l$, the BMI is transformed to:

$$I = \frac{\rho \cdot a \cdot b \cdot h}{l^2} = \frac{\rho \cdot \alpha \cdot \beta \cdot l^3}{l^2} = \gamma \cdot l \quad (\gamma = \rho \cdot \alpha \cdot \beta). \tag{7a}$$

Hence, the BMI increases proportional to the body height for all people with constant body proportions (α, β constant) and constant density ρ. This is a strange consequence. By the way, supposing more realistic body forms, the consequence is similar. Consider the function:

$$z = f(x,y) = \frac{x}{y^2} \quad (x > 0, y > 0) \tag{7b}$$

of two independent variables x, y, it has the same structure as BMI in (7). The graph of this function is a surface in the three-dimensional space. Interpreting x as weight and y as height, z is their BMI. Hence, the points (x, y, z) lie on this surface. Since weight and height are dependent in some way, certain surface regions are expected to remain unused. Looking for features with constant BMI $z = c$ the corresponding level curves on the surface are parabolas with points $(c \cdot y^2, y, c)$. The BMI values for persons with constant proportions and density lie on a certain curve within the surface having the parametric representation $(\gamma \cdot l^3, l, \gamma \cdot l)$ (see Figure 1).

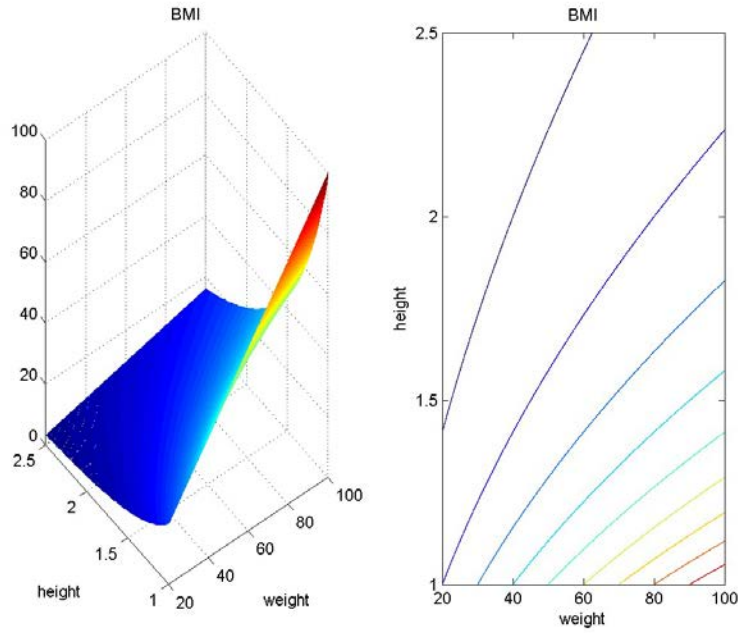

Figure 1: Surface and level curves of function (7b) in the domain [20, 100] x [1, 2.5].

PROBABILITIES

There are a lot of paradoxes in the field of probabilities, perhaps more than in any other scientific discipline [1][2]. Some paradoxes are quite simple to understand, but common sense can struggle against accepting the mathematical results (e.g. Monty Hall paradox, Prisoners' paradox).

The arithmetic average is a statistical measure. It estimates the expectation value of a probability distribution from a random sample. Often people think that average characterises the typical. So, people compare average income in their country to other countries. The country is rich if the average income is high and poor if it is low. But the average income is not at all typical, e.g. if a country has some very rich people and a lot of very poor people or if middle incomes are very rare. But nowadays, the distributions are often not unimodal (as per the Gaussian distribution). Figure 2 shows a bimodal distribution. The already mentioned scientist Quételet made a study about average humans, which caused vehement debates. This concept is problematic because, among other things, averages of single characteristics, such as height and weight do not correspond. Other people have even claimed the average is the source of beauty.

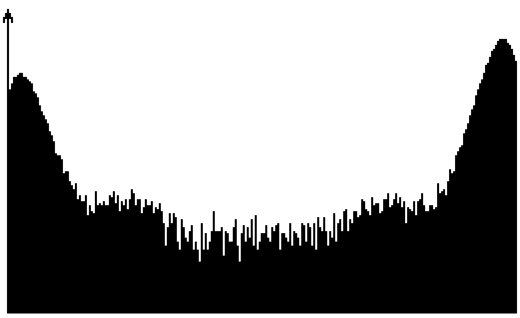

Figure 2: Bimodal noisy distribution.

It is asserted in the March 2009 edition of the American magazine, *Wired,* that a mathematical formula caused Wall Street to collapse and led to the deep finance crisis in the whole world.

$$P(T_A < 1, T_B < 1) = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma) \tag{8}$$

This formula was derived by David X. Li to estimate the risk to finance institutions of investments in correlated securities. The formula gives the probability P that enterprises A and B fail simultaneously. The formula can also be extended to more than two enterprises. Li used parallels with life insurance to calculate the probability that married couples will die in the same year. It was simple to apply this formula and finance managers used it widely. But the formula contains a parameter, the correlation coefficient γ. Li assumed this coefficient to be constant and estimated it on the basis of historical data. But at some stage, this coefficient started to become a rapidly increasing time function. The classical model failed under the new conditions. So-called finance experts used mathematics without understanding.

CONCLUSIONS

The essential findings concerning curriculum, competencies and critical thinking in mathematical education of engineering students are:

- It is especially important to know the basics (basic facts as well as basic techniques) because they are used in all mathematical disciplines and in practical applications. Besides, these basics depend little on the historical development. Poor knowledge of the basics reduces the chance of getting a satisfactory job.
- It can be more important to learn the kind of thinking required by mathematics (the methodology) than to learn the solution of certain time-dependent problem classes.
- Modelling should be an essential part of the curriculum, since it is necessary to understand the use of mathematics in engineering theory and in practice. Simple models can be useful in showing what can happen and why.
- A reasonable curriculum should supply not only problem-solving competencies for certain disciplines but also for everyday life by making general processes more transparent. Hence, mathematical competencies make an important contribution to the welfare of our society.
- Mathematical educators should encourage young people to act not only emotionally to societal developments but also rationally using mathematical or logical arguments.

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