Modelling of oscillators: general framework and simulation projects

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ABSTRACT: Oscillations play an important part in life and in engineering practice. The simplest oscillations are harmonic, represented by sine or cosine time functions and described by a second order linear differential equation with constant coefficients. Well-known examples are a spring oscillator or a pendulum for small displacements. But most oscillations are enharmonic, described by nonlinear differential equations or a system of differential equations. The solution of these equations requires the use of numerical methods implemented in mathematical computer software such as MATLAB. The software can produce simulations of real oscillations. On the basis of these simulations, the properties of the oscillating system can be predicted. Some surprising effects can also be studied. The degree of coincidence between real behaviour and the simulation allows conclusions to be drawn about the quality of the model. Presented in this article are the theoretical background and simulation projects concerning interesting oscillators. The material can be used for teaching units, combining mathematical, physical and software knowledge.

Keywords: mathematics for engineers, oscillator models, modelling and simulation, projects for students

INTRODUCTION

The world is complex. But engineering education often presents knowledge about the world in different ways. For applications, this accumulation of knowledge divided into different areas must be brought together. So a natural idea was to create interdisciplinary projects for engineering students. Additionally, projects were very suitable for co-operation between staff members and students [1].

Basic knowledge in mathematics and physics is often necessary to the understanding of engineering processes [2]. In addition, computer science and computer software are becoming even more important to the handling of complex phenomena and the relations between them. Projects about oscillators are presented requiring knowledge in these disciplines. The mathematical point of view is especially stressed. Starting with simple oscillators and models, the difficulty and complexity can be increased depending on the teaching aims and ability of the students. Much suitable material is presented in the books [3] and [4].

MODELS

The benefits and difficulties of modelling and simulation are well known [1]. At first, for the sake of simplicity, oscillators with one degree of freedom (single-DOF) were considered. The modelling starts with a physical law, e.g. Newton's law:

$$F_a = m \cdot a = m \cdot x''(t), \tag{1}$$

where m is the mass, $x = x(t)$ is the displacement (from a neutral position or equilibrium) in time t and $a = x''(t)$ is the acceleration of the oscillator. The resulting inertial force F_a is balanced by other forces, e.g. restoring force F_c, damping force F_d and external force F_e depending on its influence in the physical model.

A very simple and important model is the *harmonic oscillator*, which is fundamental both in mechanical and electrical engineering. On the one hand, the behaviour is well known. On the other hand, only under some restrictions the oscillator's behaviour is approximately harmonic. Considering viscous (velocity proportional) damping and excitation the model is:

$$F_a + F_d + F_c = m x''(t) + d\, x'(t) + c\, x(t) = F_e(t), \quad x(0) = x_0, \quad x'(0) = v_0. \tag{2}$$

The equation contains parameters (mass $m > 0$, damping coefficient $d \geq 0$, stiffness coefficient $c > 0$). The excitation force can be also a harmonic function with additional parameters (amplitude A, circular frequency ω, phase shift φ):

$$F_e(t) = A \sin(\omega\, t + \varphi). \tag{3}$$

The standard example for model (2) is a *spring-mass oscillator* (see Figure 1). But the model holds approximately also for a *gravity pendulum* if the angular displacements are small (see Figure 2).

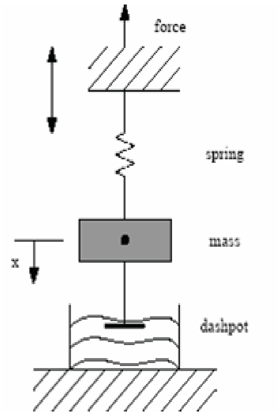

Figure 1: Spring-mass oscillator.

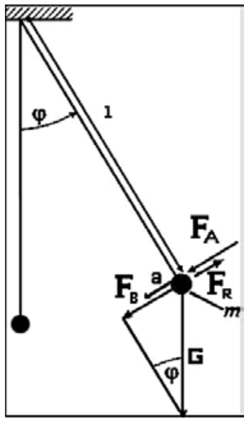

Figure 2: Gravity pendulum.

From a mathematical point of view, Model (2) is a linear differential equation of second order with constant coefficients accompanied by two initial conditions (*initial value problem*). The conditions ensure a unique solution. The solution is well known. Nevertheless, a lot of experiments can be done to demonstrate its essential features.

First consider *free motion* ($F_e = 0$, homogeneous equation). If damping is neglected ($d = 0$, undamped free motion), then there is a *harmonic oscillation* with natural circular frequency ω_0 and time period T_0:

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t), \quad \omega_0 = \sqrt{\frac{c}{m}}, \quad T_0 = \frac{2\pi}{\omega_0}. \tag{4}$$

If viscous damping is considered ($d > 0$), three different solution cases occur depending on the value of d :

1. *underdamping* for small d : exponentially decreasing oscillation;
2. *overdamping* for great d : exponentially decreasing aperiodic motion;
3. *critical damping*: decreasing aperiodic motion (limiting case between 1 and 2).

Now turn to *forced motion*. It can be decomposed into a *steady-state* response (particular solution) and a *transient* response (solution of the homogeneous equation). The transient part tends to vanish if damping is present. If the excitation force F_e is harmonic, *resonance* effects are important if the excitation and natural frequencies are nearly the same ($\omega \approx \omega_0$). In this case, the oscillation amplitudes can increase dangerously. Suitable damping can help to suppress such effects.

Often the oscillators are nonlinear. For example, the restoring force of the gravity pendulum is $F_c = c \cdot \sin(x(t))$ instead of $F_c = c \cdot x(t)$ in the linear case (see Figure 2). The damping law depends on the medium and on the velocity of the moving oscillator in this medium. The force for the so-called *turbulent damping* is $F_d = d \cdot \mathrm{sign}(x(t)) \cdot x'^2(t)$ instead of $F_d = d \cdot x'(t)$. Another important case is *Coulomb damping*. A more general class of models is:

$$F_a + F_d + F_c = m \cdot x''(t) + Q(x'(t)) + P(x(t)) = F_e(t), \quad x(0) = x_0, \quad x'(0) = v_0. \tag{5}$$

Here Q and P are arbitrary functions of a variable describing the type of damping and restoring forces. The gravity pendulum, oscillators with nonlinear hardening springs and the Duffing oscillator (Example 2) are governed by such a model. Nevertheless, in this case, the inner forces of the oscillator superpose. But, there are oscillators where this is not true. This leads to the general model scheme:

$$m \cdot x''(t) + R(x(t), x'(t)) = F_e(t), \quad x(0) = x_0, \quad x'(0) = v_0. \tag{6}$$

Here R is an arbitrary function of two variables (see Example 5 below).

There are some special effects which can arise in the nonlinear case. Generally the oscillations are enharmonic and asymmetric or even chaotic, where time periods depend on the initial conditions. There can be several stable and unstable equilibrium positions of different types.

If a solution (law of motion) $x(t)$ is known, the velocity $v(t) = x'(t)$ and the acceleration $a(t) = x''(t)$ can be calculated. Considering the oscillator in the framework of dynamics using the state space (phase), the velocity is second magnitude. With the notation $z_1(t) = x(t)$ and $z_2(t) = x'(t)$ the equation of second order is rewritten as a system of first order:

$$z_1'(t) = z_2(t), \quad z_2'(t) = -\frac{1}{m} \cdot R(z_1(t), z_2(t)) + \frac{1}{m} \cdot F_e(t), \quad z_1(0) = x_0, \quad z_2(0) = v_0. \tag{7}$$

This form is necessary if MATLAB is used for a numerical solution. Besides, *phases* can be studied by considering the relation between z_1 and z_2 (see the section below). This representation opens new horizons for modelling. Using vector notation the Model (7) fits into the model scheme:

$$\mathbf{z}'(t) = \mathbf{R}(\mathbf{z}(t)) + \mathbf{F}(t), \quad \mathbf{z}(0) = \mathbf{z}_0. \tag{8}$$

Here $\mathbf{z}$ and $\mathbf{z}_0$ are vectors of n coordinates, $\mathbf{R}$ is a vector function mapping vectors into vectors and $\mathbf{F}$ is a vector-valued function mapping numbers into vectors. Now, oscillators satisfying higher order differential equations and oscillator systems consisting of some or many interacting oscillators can be represented. Further, oscillating systems in nature are also included (e.g. predator-prey populations).

NUMERICAL METHODS

Finite difference method can be used for model problem (6) by replacing the derivatives (differential quotients) by difference quotients for discrete equidistant times t_i $(i = 0, ..., n)$ as follows:

$$x'(t) \rightarrow \frac{x(t_{i+1}) - x(t_i)}{\Delta t}, \quad x''(t) \rightarrow \frac{x(t_{i+1}) - 2x(t_i) + x(t_{i-1})}{(\Delta t)^2}. \tag{9}$$

Then, a staggered system of equations arises that allows the determination of approximate values $\tilde{x}_i \approx x(t_i)$ step by step. For free pendulum motion this is demonstrated in [1].

The simple Euler integration method applied to model problem (8) reads:

$$\mathbf{z}(t_{i+1}) = \mathbf{z}(t_i) + \mathbf{R}(\mathbf{z}(t_i))\Delta t + \mathbf{F}(t_i)\Delta t \quad (i = 0, ..., n-1), \quad t_0 = 0, \quad t_n = t^*, \quad \Delta t = \frac{t^*}{n}. \tag{10}$$

This calculates approximate vector values in the time interval $[0, t^*]$. Because of error propagation, such simple methods should be replaced by more sophisticated methods, such as Runge-Kutta. Moreover, for stiff model problems, where solutions contain parts with quite different declining behaviour, some special methods have been developed.

Checks of the numerical results using knowledge of qualitative behaviour should be performed whenever possible. During the modelled time interval the errors will generally increase. Numerical methods can lead to artificial damping or an amplification of oscillations. Periodical motions lead to almost periodic motions in numerical simulations, provided appropriate methods are used.

A too-coarse discretisation can result in monotone decreasing or increasing time functions instead of periodic ones. Nevertheless, beginners can easily understand the simple methods and gain some insight into the behaviour of the oscillator, at least for standard cases. Comparing the results of the simple and the more sophisticated methods gives some feeling about the quality of such methods, especially for the more critical examples.

PHASES AND STATE SPACE

Oscillators may be in the rest position or oscillate around, or abruptly go to the rest position, or move away from the rest position, or behave chaotically. It is important to determine their qualitative behaviour. Considering trajectories $\mathbf{z}(t)$ in the state space, the rest positions can be detected by putting $\mathbf{z}'(t)=\mathbf{0}$ in Model (8). This leads to the system $\mathbf{R}(\mathbf{z})+\mathbf{F}=\mathbf{0}$ of equations, which may have no or several solutions $\mathbf{z}=\mathbf{z}_s$ (equilibrium or rest positions).

Often numerical methods are necessary to derive these solutions. It is important to investigate the *stability* of equilibrium positions. Roughly speaking, trajectories are attracted by a stable equilibrium and distracted by an unstable equilibrium.

In the case of an oscillator with state vector $\mathbf{z}(t)=(z_1(t),z_2(t))=(x(t),x'(t))$ phase plots with axes z_1 and z_2 are useful. The rest condition is $z_1'=z_2'=0$. Representing motions as functional relations $\Phi(z_1,z_2)=0$ *phase curves* (trajectories) arise for fixed initial conditions. This gives *phase portraits* (families of trajectories) for variable initial conditions.

Free undamped motions are periodic and have closed trajectories around the centres (rest positions), if the energy level is not too high. Free damped motions are caught earlier or later by spiral sinks (see Figure 3).

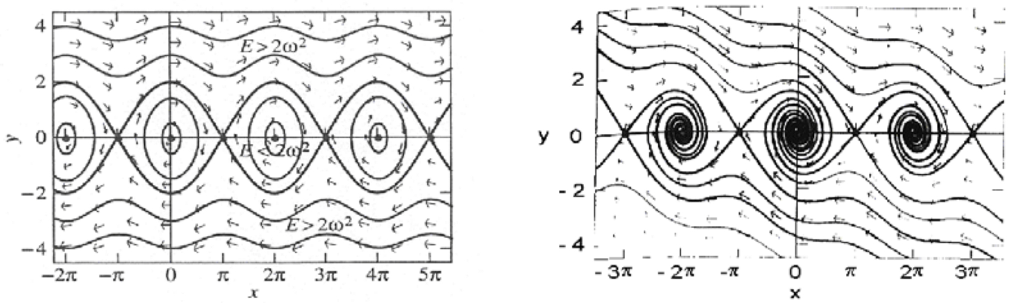

Figure 3: Phase portraits of gravity pendulum a) undamped, b) viscously damped.

The phase concept suggests considering the mechanical *energy* of oscillators. For free undamped motion, the energy split into the potential and the kinetic components is constant and equal to the initial energy E_0 depending on the initial values. This is a result of the law of conservation of energy. For free damped motion, the total energy decreases (dissipation). For Model (5) the energy relation reads in state variables:

$$E(z_1,z_2)=E_{pot}(z_1)+E_{kin}(z_2)=\int P(z_1)\,dz_1+\frac{m}{2}z_2^2=E_0-\int Q(z_2)\,z_2\,dt. \qquad (11)$$

This corresponds to phases $\Phi(z_1,z_2)=0$. Excitation changes the total energy. Considering the oscillator system as a dynamical system, the model class (8) can be modified to:

$$\mathbf{z}'(t)=\mathbf{S}(\mathbf{z}(t),\mathbf{F}_e(t),\mathbf{p}(t)),\quad \mathbf{y}(t)=\mathbf{V}(\mathbf{z}(t),\mathbf{F}_e(t),\mathbf{p}(t)). \qquad (12)$$

Here $\mathbf{z}$ denotes the *state* vector (e.g. displacements and velocities), the *input* vector $\mathbf{F}_e$ (e.g. external forces or constraints), the *parameter* vector $\mathbf{p}$ (e.g. stiffness and damping coefficients) and the output vector $\mathbf{y}$ (e.g. accelerations and reactive forces).

New aspects arise by integrating overall initial values, which allow a more general influence of inputs, stressing the parameters under consideration and introducing other values of interest. By suitably changing the variable magnitudes (initial values, parameters and inputs), the motion can be controlled. *Inverse problems* also can be investigated, i.e. determine the inputs if the motion or the type of motion is given.

PROJECTS

Example 1: A forced gravity pendulum with mass m, length l and damping coefficient d is considered. Denoting the angular displacement from vertical rest position by φ the model equation reads:

$$m\varphi''+d\,\varphi'+c\sin\varphi=F_e. \qquad (13)$$

Here great initial displacement or great initial velocity lead to high energy levels, with great deviation from the harmonic oscillation. Both enharmonic periodic oscillations and revolutions are possible for the free undamped model. If moderate damping is assumed, the oscillations decrease. For periodic excitation also chaotic behaviour can occur.

Example 2: For a viscously damped plate spring with mass m, damping coefficient d and external force F_e, the displacement x satisfies the equation:

$$m\,x'' + d\,x' + x^3 - x = F_e(t). \tag{14}$$

This type is called a *Duffing oscillator* (cubic restoring force). It has three rest positions, where two are stable and one is unstable. Such an oscillator type also can be realised by an iron rod of mass m fixed between two permanent magnets or by electronic flip-flops.

An external force F_e can be generated in the first case by moving the suspension point of the rod. If this force is harmonic, the motion changes from time to time, from the region of one stable rest position to the other. Under certain conditions, the motion is chaotic.

Example 3: An oscillator of the type:

$$x'' + d\,(x^2 - 1)\,x' + c\,x = 0 \tag{15}$$

is named after Van der Pol. Here, extreme nonlinear damping occurs. Observe that the factor of x' can be negative. All trajectories tend to a fixed curve (limit cycle, attractor). The voltage of certain electrical oscillation generators can be transformed into a Van der Pol equation.

Example 4: A ball is confined by two springs between walls. The ball is displaced vertically (see Figure 4). After release, vertical oscillations appear which are assumed to be viscously damped. After transforming into dimensionless magnitudes, the model equation has the form:

$$y'' + p\,y' + 2\left(1 - \frac{l}{\sqrt{y^2+1}}\right)y + q = 0, \quad y(0) = y_0, \quad y'(0) = 0. \tag{16}$$

The equilibrium states y_s can be determined by putting $y' = y'' = 0$ in the differential equation. In certain cases, three equilibrium states appear where the middle is unstable and the other two are stable (possible end positions). There are critical initial conditions which do not allow a prediction of the end position.

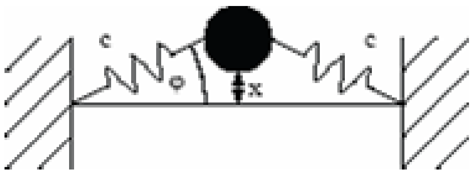

Figure 4: Vertical displacement of spring oscillator.

Example 5: A small ball with mass m is placed in a half-circular channel with radius R and released in a horizontal position with initial velocity 0. Gliding friction is considered (coefficient μ). The earth's acceleration is denoted by g. If φ is the angle of deviation from horizontal position, the law of motion is:

$$R\varphi'' + \mu\,(R\,\varphi'^2 + \mathrm{g}\sin\varphi)\,\mathrm{sign}(\varphi') - \mathrm{g}\cos\varphi = 0, \quad \varphi(0) = 0, \quad \varphi'(0) = 0. \tag{17}$$

Only a numerical solution is possible. If gliding friction is neglected ($\mu = 0$), the problem is similar to the pendulum problem (Example 1).

Example 6: A simple model of a car suspension considers only a quarter of the body (frame, assembly, motor gear) with mass m_a, propped up by an elastic spring with stiffness c_a and a damper (shock absorber) with coefficient d_a (on one wheel) with mass m_r and stiffness c_r to model the elasticity of the tyre.

One denotes by x_a, x_r, x_s the coordinates of the body, wheel and height of the variable ground (road profile). All co-ordinates are related to the static start position. Hence, the gravity force does not occur explicitly (Figure 5a). The equations of motion are:

$$\begin{aligned} m_a\,x_a'' + d_a x_a' - d_a\,x_r' + c_a\,x_a - \qquad c_a\,x_r &= 0 \\ m_r\,x_r'' - d_a x_a' + d_a\,x_r' - c_a\,x_a + (c_a + c_r)\;x_r &= c_r\,x_s \end{aligned} \tag{18}$$

So, a viscous damped linear two-mass oscillator arises. Using the first part of state space formulation (12) one obtains the following equation:

$$\begin{pmatrix} x_a' \\ x_a'' \\ x_r' \\ x_r'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{c_a}{m_a} & -\frac{d_a}{m_a} & \frac{c_a}{m_a} & \frac{d_a}{m_a} \\ 0 & 0 & 0 & 1 \\ \frac{c_a}{m_r} & \frac{d_a}{m_r} & -\frac{c_a + c_r}{m_r} & -\frac{d_a}{m_r} \end{pmatrix} \cdot \begin{pmatrix} x_a \\ x_a' \\ x_r \\ x_r' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{c_r}{m_r} \end{pmatrix} \cdot x_s \tag{19}$$

It is also useful to consider the car body's acceleration x_a'' and the dynamical wheel load F_r. Then, the quality of the car suspension can be evaluated. Hence, the second part of (12) reads:

$$\begin{pmatrix} x_a'' \\ F_r \end{pmatrix} = \begin{pmatrix} -\frac{c_a}{m_a} & -\frac{d_a}{m_a} & \frac{c_a}{m_a} & \frac{d_a}{m_a} \\ 0 & 0 & -\frac{c_r}{m_r} & 0 \end{pmatrix} \cdot \begin{pmatrix} x_a \\ x_a' \\ x_r \\ x_r' \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{c_r}{m_r} \end{pmatrix} \cdot x_s \tag{20}$$

Natural initial conditions are $x_a(0) = x_a'(0) = x_r(0) = x_r'(0) = 0$. The influence of a ramp with height h and length l can be investigated, for instance, under the assumption that the car has constant velocity v_0. Road waves can be modelled by using trigonometric time functions. If the road profile is more complicated, numerical methods are useful. A MATLAB file is given in paper [5]. Some results illustrated by graphical representations are contained in Reference [6]. A car model with chassis and four wheels is more realistic. But then, a lot of additional variables have to be considered (Figure 5b). If the restoring and damping forces are chosen to be nonlinear, numerical methods are necessary. Finally, movies of the car motion can be made.

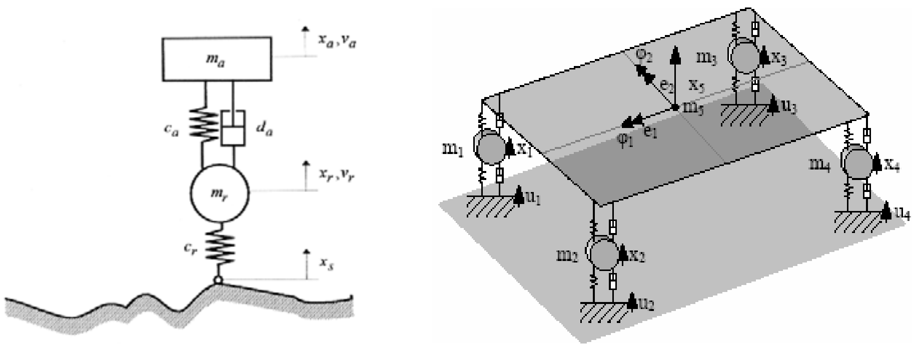

Figure 5: Suspension of car motion; a) model with one wheel, b) model with four wheels.

REALISATION OF THE PROJECTS

In the first semesters of an engineering study, often there is more focus on teaching than solving practical problems. In this case the following steps are proposed for organising and undertaking projects:

- Choose a suitable oscillator model class that is interesting from the point of teaching the theory, or has practical relevance.
- Provide information about special oscillators that fit into this class. Study the modelling process. Pay attention to the model assumptions.
- Start with simple oscillators in the class to obtain some rough feeling for the principal behaviour. Incorporate various physical magnitudes associated with the process, such as time periods, velocities, reactive forces, energy. This can help to understand the evaluation.
- Look for suitable solution methods and for suitable software. Develop simulation routines and test their correctness. Think about the user interface and geometric presentation of the simulation results.
- Change the parameters and initial values in the model to study different scenarios. Formulate the findings. Try to optimise parameters or to modify the model with respect to given goals.
- Look for theoretical results referring to the considered oscillators. Compare with the simulation results.
- Study the influence of numerical effects in the solution process. Experiment with different methods and compare.
- Investigate the overall quality of the solutions. For conservative oscillators, the total energy has to be constant. For low energy, the phase curves have to be closed. This can be checked by numerical computation.

- If possible, look for real oscillators or oscillator models. Study the real systems by making experiments or by using experimental results of other authors. Compare real processes with simulations. If the deviations are great, ask for causes. Think also about changing the mathematical model.

MATLAB is an excellent tool to realise the projects presented here. Some tasks for developing software and some experiments using MATLAB or JAVA applets are also proposed in Reference [2]. A similar teaching concept for final year dynamics and vibrations is briefly described in Reference [7].

CONCLUSIONS

Modelling oscillators in the form of projects greatly motivates students. The work is interdisciplinary in teams. The mathematics used in the solution of engineering problems is clearer and, hence, better accepted. Further, students learn that there is a big difference between a real oscillator and the corresponding model. Hence, the real motion will differ from the model motion. If the model motion is generated by numerical simulation, further effects occur. A careful analysis is necessary to obtain the correct results. This requires solid mathematical knowledge [1].

MATLAB offers a large number of professional tools to implement solutions. However, students have to develop their own routines for the software package to be used for tests and experiments, as well as for the presentation and evaluation of solutions.

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BIOGRAPHY

Dieter Schott is a Professor of Numerical Mathematics and Technical Mechanics in the Department of Electrical Engineering and Computer Science at Hochschule Wismar – University of Technology, Business and Design, Wismar, Germany. He graduated from the University of Rostock, Germany, as a mathematician in 1972. He there received his Doctorate in 1976 and the Habilitated Doctor's degree in 1982 in the field of Mathematics. Later, he worked at the Universities of Güstrow and Rostock. There, he was engaged in the education of scientists, teachers and engineers. His numerous publications are mainly related to the field of numerical analysis.

Professor Schott promotes the use of computers in lectures on basic sciences, and also promotes the introduction of international degree programmes. He has been Director of the *Gottlob-Frege-Centre* at Hochschule Wismar – University of Technology, Business and Design, since its foundation in November 2000. At the Centre, basic science education is co-ordinated, in order to strengthen, update and internationalise it.

Professor Schott organises regional and international conferences concerning didactics of engineering mathematics. He is the author of a textbook *Engineering Mathematics with MATLAB*.